Counter-examples for convergence to eigenvectors

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1 Question

If we have an endomorphism on a finite-dimensional vector space V over a field K represented by the matrix A with eigenvectors $v_1, v_2, ..., v_n$ (up to scalars), then does the sequence $(x_i)_{i\in\mathbb{N}} = A^i w$ converge to an eigenvector $\mu \cdot v_k$ for any vector $w \in V$, a scalar $\mu \in K$ and an index $k \in \{1, ..., n\}$.

2 Counter-example 1

Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ then the sequence $(x_i)_{i\in\mathbb{N}} = A^i w$ is given by $x_j = \begin{pmatrix} j+1 \\ 1 \\ 1 \end{pmatrix}$ as can be proven by induction[1]. However, the only eigenvalue of A is 1 as the characteristic polynomial is $p_A(\lambda) = (\lambda - 1)^3$ and the Eigenraum to this eigenvalues is $Eig_1(A) = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$.

In particular, there does not exist any vector $w' \in Eig_1(A)$ with $(x_i)_{i\in\mathbb{N}} \xrightarrow{i\to\infty} w'$ because for all $w' = \begin{pmatrix} w'_1 \\ w'_2 \\ w'_3 \end{pmatrix} \in Eig_1(A)$ we have that the second component $w'_2 = 0$, since $w' = \alpha \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ for some $\alpha, \beta \in K$. However, it also holds that for all $i \in \mathbb{N}$ the second component of x_i equals 1 so $(x_i)_{i\in\mathbb{N}}$ does not converge to any $w' \in Eig_1(A)$, since the second component doesn't.

3 Adapted Question

Does the statement at least hold for endomorphism induced by diagonalisable or even diagonal matrices?

4 Counter-example 2

No, not even this holds, as the following counter example shows:

Let $A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ which is a diagonal matrix and, hence, in particular diagonalisable. The matrix' eigenvalues are $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ as $p_A(\lambda) = (\lambda - 3)(\lambda - 2)(\lambda - 1)$. It's Eigenräume are given by $Eig_1(A) = \langle \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \rangle$, $Eig_2(A) = \langle \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \langle$, $Eig_3(A) = \langle \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \rangle$. Now consider $w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

[1] see Appendix

Then the sequence $(x_i)_{i\in\mathbb{N}} = (A^i w)_{i\in\mathbb{N}}$ has the elements $x_i = \begin{pmatrix} 3^i \\ 2^i \\ 1 \end{pmatrix}$ as can be shown by induction.

However, $x_i)_{i\in\mathbb{N}}$ clearly does not converge to any vector in $Eig_2(A)$ or $Eig_3(A)$ for similar reasons as in counter-example 1 for the third component. Also, $x_i)_{i\in\mathbb{N}}$ does not converge to any vector in $Eig_1(A)$ because the first component of x_i tends to ∞ as $i \to \infty$, while the first component of any vector in $Eig_1(A)$ equals 0.

5 Appendix: Induction

We prove that $(x_i)_{i\in\mathbb{N}} = A^i w$ is given by $x_j = \begin{pmatrix} j+1 \\ 1 \\ 1 \end{pmatrix}$.

Proof.

IV (Induktionsvoraussetzung):

For $j = 0$ we have $x_0 = w = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} j+1 \\ 1 \\ 1 \end{pmatrix}$.

IA (Induktionsannahme):

$\forall k \in \mathbb{N} : k \leq j : x_k = \begin{pmatrix} k+1 \\ 1 \\ 1 \end{pmatrix}$.

IS (Induktionsschritt):

For x_j we have $x_j = A^j w = A(A^{j-1} w) = A x_{j-1} \overset{IA}{=} A \begin{pmatrix} j-1+1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} j \\ 1 \\ 1 \end{pmatrix} =$ $\begin{pmatrix} j+1 \\ 1 \\ 1 \end{pmatrix}$ as we wanted to show.

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